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2020학년도 9월 평가원(가형) 30번

실수 전체의 집합에서 미분가능한 함수 $f(x)$가 모든 실수 x에 대하여

$$f'(x^2+x+1)=\pi f(1)\sin\pi x+f(3)x+5x^2$$

을 만족시킬 때, $f(7)$의 값을 구하시오. [4점] 93

sol1)

$$(2x+1)f'(x^2+x+1)=f(1)(2x+1)\pi\sin(\pi x)+f(3)(2x^2+x)+10x^3+5x^2$$

$$\downarrow \int$$

$$f(x^2+x+1)=f(3)\left(\frac{2}{3}x^3+\frac{1}{2}x^2\right)+\frac{5}{2}x^4+\frac{5}{3}x^3-f(1)(2x+1)\cos(\pi x)+\frac{2}{\pi}f(1)\sin(\pi x)+C$$

$x=0$ 대입 $\longrightarrow C=2f(1)$

$x=1$ 대입 $\longrightarrow f(3)=\frac{7}{6}f(3)+\frac{25}{6}+3f(1)+2f(1)$

$x=-1$ 대입 $\longrightarrow f(1)=-\frac{1}{6}f(3)+\frac{5}{6}-f(1)+2f(1)$

$\longrightarrow f(1)=-1,\ f(3)=5$

$\therefore f(x^2+x+1)=\frac{5}{2}x^4+5x^3+\frac{5}{2}x^2+(2x+1)\cos(\pi x)-\frac{2}{\pi}\sin(\pi x)-2$

$x=2$ 대입 $\longrightarrow f(7)=93$

sol2) 역함수 이용, $f(x)$ 구하기

$p(x)=x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\quad\left(x\geq-\frac{1}{2}\right)$

$g(x)=x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\quad\left(x<-\frac{1}{2}\right)$

$\longrightarrow p(x),\ g(x)\geq\frac{3}{4}$

$g(x)=p^{-1}(x)=\sqrt{x-\frac{3}{4}}-\frac{1}{2}\quad\left(x\geq\frac{3}{4}\right)$

$h(x)=g^{-1}(x)=-\sqrt{x-\frac{3}{4}}-\frac{1}{2}\quad\left(x\geq\frac{3}{4}\right)$

i) $x \ge -\frac{1}{2}$

$f'(p(x)) = \pi f(1)\sin(\pi x) + f(3)x + 5x^2$

$x = g(t)$ 대입

$f'(t) = \pi f(1)\sin(\pi g(t)) + f(3)g(t) + 5\{g(t)\}^2$

$f'(t)g'(t) = \pi f(1)\sin(\pi g(t))g'(t) + f(3)g(t)g'(t) + 5\{g(t)\}^2 g'(t)$

$g'(t) = \dfrac{1}{2\sqrt{x-\frac{3}{4}}} = \dfrac{1}{2g(t)+1}$

$f'(t) = \pi f(1)\sin(\pi g(t))g'(t)\cdot\{2g(t)+1\} + f(3)g(t)g'(t)\{2g(t)+1\} + 5\{g(t)\}^2 g'(t)\{2g(t)+1\}$

$\downarrow \int$

$f(t) = -f(1)\{2g(t)+1\}\cos(\pi g(t)) + f(1)\frac{2}{\pi}\sin(\pi g(t)) + f(3)\left\{\frac{2}{3}\{g(t)\}^3 + \frac{1}{2}\{g(t)\}^2\right\}$
$+ 5\left\{\frac{1}{2}\{g(t)\}^4 + \frac{1}{3}\{g(t)\}^3\right\} + C_1$

$t=1$ 대입 $\longrightarrow f(1) = -f(1) + C_1$

$t=3$ 대입 $\longrightarrow f(3) = 3f(1) + \frac{7}{6}f(3) + \frac{25}{6} + C_1$

ii) $x < -\frac{1}{2}$

$f'(g(x)) = \pi f(1)\sin(\pi x) + f(3)x + 5x^2$

$x = h(t)$ 대입

$f'(t) = \pi f(1)\sin(\pi h(t)) + f(3)h(t) + 5\{h(t)\}^2$

$f'(t)h'(t) = \pi f(1)\sin(\pi h(t))h'(t) + f(3)h(t)h'(t) + 5\{h(t)\}^2 h'(t)$

$h'(t) = \dfrac{1}{2\sqrt{x-\frac{3}{4}}} = \dfrac{1}{2h(t)+1}$

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$$f'(t) = \pi f(1)\sin(\pi h(t))h'(t)\cdot\{2h(t)+1\} + f(3)h(t)h'(t)\{2h(t)+1\} + 5\{h(t)\}^2 h'(t)\{2h(t)+1\}$$

$$\Big\downarrow \int$$

$$f(t) = -f(1)\{2h(t)+1\}\cos(\pi h(t)) + f(1)\frac{2}{\pi}\sin(\pi h(t)) + f(3)\left\{\frac{2}{3}\{h(t)\}^3 + \frac{1}{2}\{h(t)\}^2\right\}$$
$$+ 5\left\{\frac{1}{2}\{h(t)\}^4 + \frac{1}{3}\{h(t)\}^3\right\} + C_2$$

$t=1$ 대입 $\longrightarrow f(1) = -f(1) - \frac{1}{6}f(3) + \frac{5}{6} + C_2$

f는 연속이므로 $x=\frac{3}{4}$에서 연속이다.

$\therefore C_1 = C_2,\ f(1) = -1,\ f(3) = 5,\ C_1 = C_2 = -2$

$\therefore f(t) = \{2g(t)+1\}\cos(\pi g(t)) - \frac{2}{\pi}\sin(\pi g(t)) + \frac{5}{2}\{g(t)\}^2\{g(t)+1\} - 2 \quad \left(t \geq \frac{3}{4}\right)$

$\therefore f(x) = \sqrt{4x-3}\sin\left(\pi\sqrt{x-\frac{3}{4}}\right) + \frac{2}{\pi}\cos\left(\pi\sqrt{x-\frac{3}{4}}\right) + \frac{5}{2}(x-1)^2 - 2 \left(x \geq \frac{3}{4}\right)$

$\therefore f(7) = 93$